

ANOTHER PROOF OF A LEMMA BY L. SHEPP

TOMAS PERSSON

ABSTRACT. We give a new proof of a lemma by L. Shepp, that was used in connection to random coverings of a circle.

Consider a circle of circumference 1, and a sequence l_n of number in $(0, 1)$. We try to cover the circle by tossing arcs of length l_n on the circle. In [2], L. Shepp proved that if the arcs are tossed independently and uniformly distributed on the circle, then the circle is covered with probability one, if and only if $\sum n^{-2} e^{l_1 + \dots + l_n}$ diverges. In the proof of this result, Shepp used the following lemma.

Lemma 1. *Let l_n be a decreasing sequence of numbers in $(0, 1)$, and $0 < \varepsilon < 1 - l_1$. Suppose that $\sum l_n^2$ diverges. Then*

$$\limsup_{n \rightarrow \infty} \int_0^\varepsilon \prod_{k=1}^n \frac{1 - l_k - \min\{l_k, t\}}{(1 - l_k)^2} dt = \infty.$$

Shepp's proof of this lemma is based on some related considerations of probabilities, and he wrote that "It seems difficult to prove directly that $\sum l_n^2 = \infty$ implies that ... holds." This induced T. Kaijser to look for a simple and direct proof, which indeed he found in [1]. In this note we provide yet another proof. In fact, we shall prove the somewhat stronger statement that

$$(1) \quad \lim_{n \rightarrow \infty} \int_0^\varepsilon \prod_{k=1}^n \frac{1 - l_k - \min\{l_k, t\}}{(1 - l_k)^2} dt = \infty.$$

We will make use of the following sometimes very useful inequality. It might be well-known to the reader, but we provide nevertheless a proof.

Lemma 2. *Let f_1, \dots, f_n be positive functions, that are either all increasing or all decreasing. Then*

$$\varepsilon^{n-1} \int_0^\varepsilon \prod_{k=1}^n f_k(x) dx \geq \prod_{k=1}^n \int_0^\varepsilon f_k(x) dx.$$

Proof. We have that

$$f_1(x) \geq f_1(y) \iff f_2(x) \geq f_2(y),$$

since f_1 and f_2 are either both increasing or both decreasing. Hence

$$(f_1(x) - f_1(y))(f_2(x) - f_2(y)) \geq 0$$

Date: November 21, 2014.

2010 *Mathematics Subject Classification.* 26D15.

for all x and y , and therefore

$$\int_0^\varepsilon \int_0^\varepsilon (f_1(x) - f_1(y))(f_2(x) - f_2(y)) \, dx \, dy \geq 0.$$

This yields that

$$\varepsilon \int_0^\varepsilon f_1(x)f_2(x) \, dx \geq \int_0^\varepsilon f_1(x) \, dx \int_0^\varepsilon f_2(x) \, dx.$$

So far, we have not used that the functions are positive, but this will be used in the following step. Any product of the functions f_1, \dots, f_n is monotone, and the proof is now finished by induction. \square

We are now ready to prove (1). Put

$$f_k(t) = \frac{1 - l_k - \min\{l_k, t\}}{(1 - l_k)^2}, \quad k = 1, 2, \dots$$

When $l_k < \varepsilon$, a direct calculation shows that

$$(2) \quad \int_0^\varepsilon f_k(t) \, dt = \frac{\frac{1}{2}l_k^2 + \varepsilon - 2\varepsilon l_k}{(1 - l_k)^2}, \quad k = 1, 2, \dots$$

We consider the function

$$g_\varepsilon(x) = \frac{1}{\varepsilon} \frac{\frac{1}{2}x^2 + \varepsilon - 2\varepsilon x}{(1 - x)^2}.$$

One easily checks that $g_\varepsilon(0) = 1$, $g'_\varepsilon(0) = 0$, and $g''_\varepsilon(0) = \frac{1-2\varepsilon}{\varepsilon}$. Hence, we have that

$$(3) \quad g_\varepsilon(x) = 1 + \frac{1 - 2\varepsilon}{2\varepsilon} x^2 + o(x^2).$$

Since the functions f_1, f_2, \dots are all positive, it is sufficient to prove (1) for small ε . Hence we may and will assume that $\varepsilon < \frac{1}{2}$ so that the coefficient $\frac{1-2\varepsilon}{2\varepsilon}$ in (3) is positive.

Let m be such that $l_k < \varepsilon$ for all $k > m$. By Lemma 2 and (2) we have for any $n > m$ that

$$\int_0^\varepsilon \prod_{k=1}^n f_k(t) \, dt \geq \varepsilon^{-m} \prod_{k=1}^{m-1} \int_0^\varepsilon f_k(t) \, dt \prod_{k=m}^n g_\varepsilon(l_k) = C \exp\left(\sum_{k=m}^n \log g_\varepsilon(l_k)\right),$$

where the positive constant C does not depend on n . By (3) and $\sum_{k=1}^\infty l_k^2 = \infty$, we conclude that (1) holds.

REFERENCES

- [1] T. Kaijser, *A Note on a Theorem by Larry Shepp*, Unpublished Report, Linköping University 1978, LiTH-MAT-R-78-18c.
available at <http://liu.diva-portal.org/smash/record.jsf?pid=diva2:764680>
- [2] L. A. Shepp, *Covering the circle with random arcs*, Israel Journal of Mathematics 11 (1972), 328–345.

TOMAS PERSSON, CENTRE FOR MATHEMATICAL SCIENCES, LUND UNIVERSITY, BOX 118, 22100 LUND, SWEDEN

E-mail address: tomasp@maths.lth.se

URL: <http://www.maths.lth.se/~tomasp>